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Vibration of two layered cylindrical shells of variable thickness with fluid interaction

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ABSTRACT

Cylindrical shell of variable thickness filled with fluid interaction using spline approximation is investigated to determine the vibrational behavior of the shell. The shell is made up of two layers of isotropic or specially orthotropic materials and the thickness variation is considered. In this study, irrotational of an inviscid fluid is used. The equations of shell are coupled with fluid term. Love's first approximation theory is implemented to derive the equations of shell which are in terms of longitudinal, circumferential and transverse displacement functions. These functions are approximated using spline method, resulting in the generalized eigenvalue problem by combining the suitable boundary conditions. The thickness variations are assumed to be linear, exponential and sinusoidal along the axial direction of the cylinder. Frequency parameter and an associated eigenvector of the spline coefficients are analysed by considering various parameters such as relative layer thickness, length parameter, material properties, and coefficients of thickness variations under clamped-clamped and simply-supported-simply supported boundary conditions, respectively.

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1. INTRODUCTION

Composite and laminated materials have gained significant traction in various industries due to their unique properties such as high strength-to-weight ratio, high stiffness and corrosion resistance. It provides design flexibility and can be tailored to meet specific performance requirements. Each layer in a laminated shells or plates can be oriented in different directions, allowing engineers or designers to optimize stiffness and strength in specific directions according to the load path (Reddy, 2004).

Mathematical modeling on the vibrational behavior of the plate and shell structures has been extensively studied to determine frequencies and mode shapes. These works often employ different theoretical models such as classical theory and first order shear deformation theory. Several studies utilize various methods including analytical and finite element method (Attia et al., 2024), dynamic stiffness method (Zu and Wu, 2020) and generalized differential quadrature method (Bochkarev and Lekomtsev, 2025) to determine the vibrational behavior of the structures. In addition, frequencies of plate and shell structures are influenced by several factors like geometry (thickness, curvature, size), material properties, boundary conditions (clamped, simply supported, free) and thickness variation that determine their dynamic behavior and structural performance. Understanding these factors is crucial because it enables designers to predict behavior accurately and optimize the design.

Variable thickness also known as non-uniform thickness have a thickness that changes across their geometry. The vibrational behavior on structures with the variable thickness has been addressed by many researchers such as studies on plates, cylindrical shells and conical shells. These works often employ different theoretical models and methods. Lal and Saini (2020) focused on the vibration analysis of functionally graded circular plates of variable thickness under thermal environment by generalized differential quadrature method. Clamped and simply supported plates were considered. Results revealed that the frequencies of clamped plate are greater than the simply supported plate. Morruzi et al. (2024) used an adaptive finite element to investigate free vibration of variable thickness plates. Abdullah and Sani (2024) provided the comparative computational modal analysis of uniform and tapered plates. Finite element modal analysis was employed to determine the natural frequencies and mode shapes of plates. The results revealed that the plate with varying thickness has lower natural frequencies, and its mode shapes are more complex and asymmetric compared to the plates with uniform thickness.

Miao et al. (2022) presented a unified approach for the analysis of free vibrations of the three-layer functionally graded cylindrical shell with non-uniform thickness. The Sanders' shell theory is implemented to obtain the strain and curvature-displacement relations. Rayleigh–Ritz method and Chebyshev polynomials are employed to improve computational efficiency. El-Kaabazi and Kennedy (2012) implemented dynamic stiffness equations to investigate the variable thickness cylindrical shells under the assumptions of Donnell, Timoshenko and Flügge theories. Wittrick–Williams algorithm is used to determine natural frequencies. Results found that by increasing the thickness variation, the natural frequencies decrease for asymmetric linear and quadratic taper, but frequencies increase for symmetric taper. The generalized differential quadrature method was performed by Tornabene et al. (2017) to evaluate the vibrational behavior of FGM sandwich shells with variable thickness.

Research involving fluid interaction in variable thickness structures is relatively sparse due to the added complexity of fluid-structure coupling. Several studies have adopted Love's classical shell theory, also known as Love's first approximation, for modelling thin shell structures where shear deformation is negligible. Izyan et al. (2024) investigated the free vibration behavior of layered conical shells of variable thickness, considering the influence of fluid interaction. The study utilized Love's first approximation theory. The displacement functions were approximated using a spline method to solve the governing equations and the impact of different radial thickness variations such as linear, exponential, and sinusoidal was evaluated under clamped-clamped and simply supported boundary conditions.

Li et al. (2020) studied free vibration of the variable thickness functionally graded materials beams in fluid based on Timoshenko beam theory. The governing equations and boundary conditions are derived by using Hamilton's principle and then discretized by using differential quadrature method. In addition, Murari et al. (2023) implemented the functionally graded graphene origami-enabled auxetic metamaterial beams with variable thickness. The beam is placed vertically in fluid. Geometry of the three non-uniform beams considered which are bi-linear, bi-cubical and bi-sinusoidal. The equations are based on first order shear deformation theory and solved using differential quadrature and Bolotin's method. Moreover, Esmaeilzadehazimi et al. (2024) developed a finite element method model based on Sanders' thin shell theory to explore the dynamic instability of ring-stiffened conical shells subjected to internal flowing fluid. The study concluded that ring stiffeners significantly affect the stability of the cone under different boundary conditions. Instability in stiffened shells occurs at higher critical fluid velocities than in unstiffened shells across all boundary conditions.

Therefore, this study investigates the vibration of cylindrical shell with variable thickness in the presence of fluid. While variable thickness without fluid has been extensively studied, the literature on variable thickness with the presence of fluid interaction remains limited. Hence, this study ultimately aims to contribute to the less-explored domain of variable thickness shell structures under fluid interaction. The equations of motion are based on Love's first approximation theory. Shell is made up of two layers and the thickness variations are presented in several functions namely linear, exponential and sinusoidal along the radial direction under Clamped-Clamped (C-C) and Simply-supported Simply-supported (S-S) boundary conditions. Spline method is implemented in this study and this method is one of approximate method in solving boundary value problems (Bickley,1968). Other literatures that implemented spline method in their study to solve vibration behavior include layered cylindrical shells (Viswanathan & Navaneethakrishnan, 2003), layered truncated conical shells filled with quiescent fluid (Izyan et al., 2017) and cross-ply laminated plates (Javed et al., 2018). A generalized eigenvalue problem is solved numerically for the frequency parameter and an associated eigenvector of the spline coefficients. Frequencies with respect to relative layer thickness, length parameter, types of material, and coefficients of thickness variations are analysed.

2. MATHEMATICAL FORMULATION

A thin layered circular cylindrical shell (length ℓ, constant thickness *h*, radius *r)* is considered. Each layer is assumed to be homogeneous, linearly elastic and isotropic or specially orthotropic. The *x* coordinate of the shell is taken along the longitudinal direction, *θ* and *z* coordinate are in the circumferential and radial direction respectively. Equations of motion for cylindrical shell coupled with fluid is written as

$$\frac{\partial N_x}{\partial x}+\frac{1}{r}\frac{\partial N_{\theta x}}{\partial \theta}=\rho h\frac{\partial^2 u}{\partial t^2},\frac{\partial N_{x\theta}}{\partial x}+\frac{1}{r}\frac{\partial N_\theta}{\partial \theta}+\frac{1}{r}\frac{\partial M_{x\theta}}{\partial x}+\frac{1}{r^2}\frac{\partial M_\theta}{\partial \theta}=\rho h\frac{\partial^2 v}{\partial t^2},$$
$$\frac{\partial^2 M_x}{\partial x^2}+\frac{2}{r}\frac{\partial M_{\theta x}}{\partial x\partial \theta}+\frac{1}{r^2}\frac{\partial^2 M_\theta}{\partial \theta^2}-\frac{N_\theta}{r}=\rho h\left(\frac{\partial^2 w}{\partial t^2}-\frac{p}{\rho h}\right), \tag{2.1}$$

where N_x, N_θ and $N_{x\theta}$ are the stress resultants, M_x, M_θ and $M_{x\theta}$ are the moments resultants and p is the pressure.

The fluid is assumed to be incompressible. Irrotational flow of an inviscid fluid undergoing small oscillations is expressed as wave equation. According to Zhang et al. (2001), the equation of motion of the fluid can be written in the cylindrical coordinates system (*x, θ, r*)

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}+\frac{\partial^2 p}{\partial x^2}=\frac{\partial^2 p}{c^2\partial t^2} \tag{2.2}$$

where *t* is the time, *p* is the pressure and *c* is the sound of speed of the fluid. The *x* and *θ* -coordinates are the same as those of the shell, where the *r*-coordinate is taken from the *x*-axis of the shell.

The thickness of the k^{th} layer is assumed in the form $h_k(x)=h_{0k}g(x)$, where h_{0k} is a constant thickness. In general, the thickness variation of each layer is assumed in the form $h_k(x)=h_0 g(x)$, and

$$g(x)=1+C_\ell\frac{x}{\ell}+C_e\,exp\left(\frac{x}{\ell}\right)+C_s\,sin\left(\frac{\pi x}{\ell}\right). \tag{2.3}$$

If $g(x) = 1$, then the thickness becomes uniform. Therefore, A_{ij}, B_{ij} and D_{ij} corresponding to layers of uniform thickness with superscript '*c*' can easily be obtained as $A_{ij} = A_{ij}^c g(x), B_{ij} = B_{ij}^c g(x), D_{ij} = D_{ij}^c g(x)$,

in which

$$A_{ij}^c = \sum_{k=1} \bar{Q}_{ij}^k \,(z_k - z_{k-1}), B_{ij}^c = \frac{1}{2}\sum_{k=1} \bar{Q}_{ij}^k \,(z_k^2 - z_{k-1}^2), D_{ij}^c = \frac{1}{3}\sum_{k=1} \bar{Q}_{ij}^k \,(z_k^3 - z_{k-1}^3)$$

with $i, j = 1,2,6$, z_k, z_{k-1} are boundaries of the k^{th} layer.

The displacement components u, v and w are assumed in the form of

$$u(x,t) = U(x)\cos n\,\theta e^{i\omega t}, v(x,t) = V(x)\sin n\,\theta e^{i\omega t}, w(x,t) = W(x)\cos n\,\theta e^{i\omega t}, \qquad (2.4)$$

where x is the longitudinal, θ is the rotational, ω is the angular frequency of vibration, *n* is the circumferential node number and *t* is the time.

The non-dimensional parameters are as follows

$$\begin{aligned} &L = \frac{\ell}{r}; \text{a length parameter} \\ &X = \frac{x}{\ell}; \text{a distance coordinate} \\ &\delta_k = \frac{h_k}{h}; \text{ a relative layer thickness of k-th layer} \\ &H = \frac{h}{r}; \text{the thickness parameter} \\ &\lambda = \omega\ell\sqrt{\frac{R_0}{A_{11}}}; \text{a frequency parameter} \\ &R = \frac{r}{\ell}; \text{a radius parameter} \end{aligned} \qquad (2.5)$$

Here *r* is the radius of the cylinder and *h* is the total thickness of the shell. Since only two layers is considered in this study, therefore, $\delta = \delta_1$ and $\delta_2 = 1 - \delta_1$. The thickness of the k^{th} layer of the shell is assumed in the form $h_k(X) = h_{0k} g(X)$. h_{0k} is a constant thickness. Therefore, $g(X) = 1 + C_\ell X + C_e\, exp(X) + C_s\, sin(\pi X)$. If $(C_e = C_s = 0)$, then the thickness variation becomes linear. It can be written as $C_\ell = \frac{1}{\eta} - 1$, where η is the taper ratio $\frac{h_k(0)}{h_k(1)}$. If $(C_\ell = C_e = 0)$, then the excess thickness varies exponentially. If $(C_\ell = C_e = 0)$, then the excess thickness varies sinusoidally. The thickness of the layer at X = 0 is h_{0k} for the first and third cases, but the thickness is $h_{0k}(1 + C_e)$ for the second case.

In obtaining equations of shell coupled with fluid, substituting Eq. (2.2) into stress and momentum resultants, then substituting into Eq. (2.1). Next, applying Eq. (2.4-2.5), the equations in the matrix form are obtained as follows

$$\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{2.6}$$

where $L_{ij}(i = 1,2,3; j = 1,2,3)$ are the differential operators given as follows

$$L_{11} = \frac{d^2}{dX^2} + \frac{g'}{g}\frac{d}{dX} - S_{10}\frac{n^2}{R^2} + \lambda^2,$$

$$L_{12} = \frac{n}{R}\left(S_2 + S_5\frac{1}{R}\right)\frac{g'}{g'} + \frac{n}{R}\left(S_2 + S_{10} + \frac{1}{R}(S_5 + S_{11}) + \right)\frac{d}{dX},$$

$$L_{13} = -S_4\frac{d^3}{dX^3} - S_4\frac{g'}{g}\frac{d^2}{dX^2} + \left(\frac{n^2}{R^2}(S_5 + 2S_{11}) + S_2\frac{1}{R}\right)\frac{d}{dX} + \frac{g'}{gR}\left(S_2 + S_5\frac{1}{R} + \right),$$

$$L_{21} = -\frac{n}{R}\left(S_2 + 2S_{10} + \frac{1}{R}(S_5 + 2S_{11})\right)\frac{d}{dX} - \frac{n}{R}\left(S_{10} + S_{11}\frac{1}{R}\right)\frac{g'}{g},$$

$$L_{22} = \left(S_{10} + \frac{2S_{11}}{R} + \frac{S_{12}}{R^2}\right)\frac{d^2}{dX^2} + \left(S_{10} + \frac{2S_{11}}{R} + \frac{S_{12}}{R^2}\right)\frac{d}{dX} - \frac{n^2}{R^2}\left(S_3 + \frac{2S_6}{R} + \frac{S_9}{R^2}\right) + \lambda^2,$$

$$L_{23} = \frac{n}{R}\left(2S_{11} + S_5 + \frac{2S_{12}}{R} + \frac{S_8}{R}\right)\frac{d^2}{dX^2} + \frac{n}{R}\frac{g'}{g}\left(2S_{11} + \frac{2S_{12}}{R}\right)\frac{d}{dX} - \frac{n}{R^3}\left((1 + n^2)S_6 + n^2\frac{S_9}{R}\right) - \frac{nS_3}{R^2},$$

$$L_{31} = S_4\frac{g'}{g}\frac{d^2}{dX^2} + \left(S_4\left(\frac{g'^2}{g^2} + S_{10}\frac{n^2}{R^2}\right)\frac{g''}{g'} - \frac{S_2}{R} - \frac{n^2}{R^2}(S_5 + 2S_{11}) - \lambda'^2 S_4\right)\frac{d}{dX} - 2S_{11}\frac{g'}{g}\frac{n^2}{R^2},$$

$$L_{32} = \frac{n}{R}\left(2S_{11} + S_5 + \frac{1}{R}(S_8 + 2S_{12})\right) - S_4\left(S_2 + S_{10} + \frac{1}{R}(S_5 + S_{11})\right)\frac{d^2}{dX^2} + \frac{n}{R}\frac{g'}{g}\left(2\left(S_5 + S_{11} + \frac{S_8}{R} + \frac{S_{12}}{R}\right) - S_4\left(S_2 + \frac{S_5}{R}\right)\right)\frac{d}{dX} - \frac{n}{R}\left(\frac{n^2}{R^2}\left(S_6 + \frac{S_9}{R}\right) + \frac{S_3}{R} + \frac{S_6}{R^2} - S_5\frac{g''}{g} - \frac{S_8}{R}\frac{g''}{g} + S_4\left(S_2 + \frac{S_5}{R}\right)\left(\frac{g''}{g} - \frac{g'^2}{g^2}\right)\right),$$

$$L_{33} = (S_4^2 - S_7)\frac{d^4}{dX^4} + (S_4^2 - 2S_7)\frac{g^{'}}{g}\frac{d^3}{dX^3}$$
$$+\left(\frac{2S_5}{R} - S_7\frac{g^{''}}{g} + \frac{n^2}{R^2}(4S_{12} + 2S_8) - S_4\left(\frac{n^2}{R^2}(S_5 + 2S_{11}) + \frac{S_2}{R} - S_4\left(\frac{g^{''}}{g} - \frac{g^{'2}}{g^2}\right)\right)\right)\frac{d^2}{dX^2}$$
$$+\frac{g^{'}}{g}\left(\frac{2S_5}{R} + \frac{2n^2}{R^2}(S_8 + 2S_{12}) - S_4\left(S_5\frac{n^2}{R^2} + \frac{S_2}{R}\right)\right)\frac{d}{dX}$$
$$-\left(\frac{n^2}{R^2}\left(\frac{2S_6}{R} - S_8\frac{g^{''}}{g}\right) + \frac{n^4}{R^4}S_9 + \frac{S_3}{R^2} - \frac{S_5}{R}\frac{g^{''}}{g} + S_4\left(S_5\frac{n^2}{R^2} + \frac{S_2}{R}\right)\left(\frac{g^{''}}{g} - \frac{g^{'2}}{g^2}\right)\right) + \lambda^2\left(1 + \frac{\rho_f}{\rho_s h}\frac{J_n(R)}{J_n^{'}(R)}\right).$$

with

$$S_2 = \frac{A_{12}}{A_{11}}, S_3 = \frac{A_{22}}{A_{11}}, S_4 = \frac{B_{11}}{\ell A_{11}}, S_5 = \frac{B_{12}}{\ell A_{11}}, S_6 = \frac{B_{22}}{\ell A_{11}}, S_7 = \frac{D_{11}}{\ell^2 A_{11}}, S_8 = \frac{D_{12}}{\ell^2 A_{11}},$$
$$S_9 = \frac{D_{22}}{\ell^2 A_{11}}, S_{10} = \frac{A_{66}}{A_{11}}, S_{11} = \frac{B_{66}}{\ell A_{11}}, S_{12} = \frac{D_{66}}{\ell^2 A_{11}}, \lambda^2 = \frac{R_0\omega^2}{A_{11}}, R_0 = \rho h.$$

3. METHODS OF SOLUTION

The spline approximation is a lower order approximation which yields a better accuracy than a global higher order approximation (Bickley,1968). The displacement functions $U(X), V(X)$ and $W(X)$ are approximated by cubic and quintic spline functions $U^*(X), V^*(X)$ and $W^*(X)$, respectively as follows

$$U^*(X) = \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 H(X - X_j),$$
$$V^*(X) = \sum_{i=0}^{2} c_i X^i + \sum_{j=0}^{N-1} d(X - X_j)^3 H(X - X_j), \quad (2.7)$$
$$W^*(X) = \sum_{i=0}^{4} e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^5 H(X - X_j).$$

Here, $H(X - X_j)$ is the Heaviside step function. *N* is the number of intervals in the range of $X \in [0,1]$ is divided. The points of division $X = X_s = \frac{s}{N}, (s = 0,1,2, \ldots N)$ are chosen as the knots of the splines as well as the collocation points. Imposing the condition that the differential equations given by Eq. (2.6) are satisfied by these splines at the knots, a set of $(3N+3)$ homogeneous equations into $(3N+11)$ unknown spline coefficients $a_i, b_j, c_i, d_j, e_i,\ f_j (i = 0,1,2,3,4; j = 0,1,2,\dots, N-1)$ are obtained.

The following boundary conditions are considered

i. Clamped-Clamped (C-C) (both the ends are clamped)

$$U = 0, V = 0, W = 0, \frac{dW}{dX} = 0 \text{ at } X = 0 \text{ and } X = 1.$$

ii. Simply-Supported (S-S) (both the ends are simply supported)

$$U = 0, V = 0, W = 0, M_x = 0 \text{ at } X = 0 \text{ and } X = 1.$$

By applying each of these boundary conditions separately, we can obtain 8 more equations on spline coefficients. Combining these 8 equations with the earlier $(3N+3)$ equations, we get $(3N+11)$ homogeneous equations in the same number unknowns. Therefore, a generalized eigenvalue problem can be obtained as follows

$$[M]\{q\} = \lambda^2[P]\{q\}, \tag{2.8}$$

where [M] and [P] are the square matrices, {q} is the column matrix of the eigenvector of the spline coefficients and λ is the eigenparameter.

4. RESULTS ANALYSIS

Free vibration of two layered circular cylindrical shell of variable thickness was analysed. Boundary conditions considered are Clamped-Clamped (C-C) and Simply-supported Simply-supported (S-S) boundary conditions. Two types of materials are used namely High Strength Graphite (HSG) and S-Glass Epoxy (SGE) materials (Elishakoff and Stavsky, 1979). The density of fluid used in this study is $\rho_f = 1000 \text{kg/m}^3$. Relative layer thickness, thickness variation and length variation on frequencies with different boundary conditions were analysed. The first three modes of vibration were selected for the frequencies and results were illustrated in Tables and Figures. To verify the convergence of the spline method, convergence study was conducted to determine the frequency parameters of two layered shells with fluid. From the results, the number of knots N=14 is chosen since for the next value of N, the percentage change in the values of λ is very low, the maximum being 0.3%.

The variation of frequency parameter $\lambda_m (m = 1,2,3)$ with respect to the relative thickness δ under linear variation in thickness (η=0.75), exponential variation in thickness (C_e=0.2) and sinusoidal variation in thickness (C_s=0.25) under C-C and S-S boundary conditions are shown in Fig. 1. Two layered shells are arranged in the order of HSG and SGE materials. The values of the circumferential number n=2, the ratio of the shell's constant thickness to radius H=0.02, and the ratio of the shell length to the radius L=1.5 are fixed. Frequencies under C-C boundary conditions are shown in Fig.1(a), Fig.1(b), Fig.1(c) meanwhile frequencies under S-S boundary conditions are shown in Fig.1(d), Fig.1(e), Fig.1(f). From Fig.1, at δ=0, the inner layer disappears, and the shell is homogeneous, which is made of SGE material. At δ=1 the outer layer disappears, again the shell is homogeneous, made of HSG material. Generally, as δ increase, λ_m decreases for $(m = 1,2)$ for all values of $\delta \geq 0.2$.

Then, the study was conducted for the shell under S-S boundary conditions. Results showed that the behavioral frequencies are like C-C boundary conditions, however, the values on S-S boundary conditions are lower compared to C-C boundary conditions.

Fig. 2 depicts the variation of frequencies ω on length parameter for two layered shells with the materials arranged in the order HSG-SGE with $\delta = 0.5$, $H = 0.02$ and $n = 2$ under C-C boundary conditions as well as S-S boundary conditions. The variation in thickness of layer; η=0.7, C_e=0.1 and C_s=0.2 are fixed. For C-C boundary conditions, the variation of angular frequencies ω on length parameter with variation in thickness of layers with η=0.7, C_e=0.1 and C_s=0.2 as shown in Fig. 2(a), Fig. 2(b) and Fig. 2(c), respectively. Moreover, for S-S boundary conditions, the variation of angular frequencies ω on length parameter with variation in thickness of layers with η=0.7, C_e=0.1 and C_s=0.2 as shown in Fig. 2(d), Fig. 2(e) and Fig. 2(f), respectively.

For the length of the cylinder for its vibrational behavior, the angular frequency ω is considered instead of λ. Results revealed that as L increases, ω will decrease. In the range of $0.5 < L < 0.75$, the frequencies decrease fast. The frequency decreases slowly in the range of $0.75 < L < 2$. It can be observed that all the frequencies of C-C boundary conditions will give higher values compared to S-S boundary conditions.

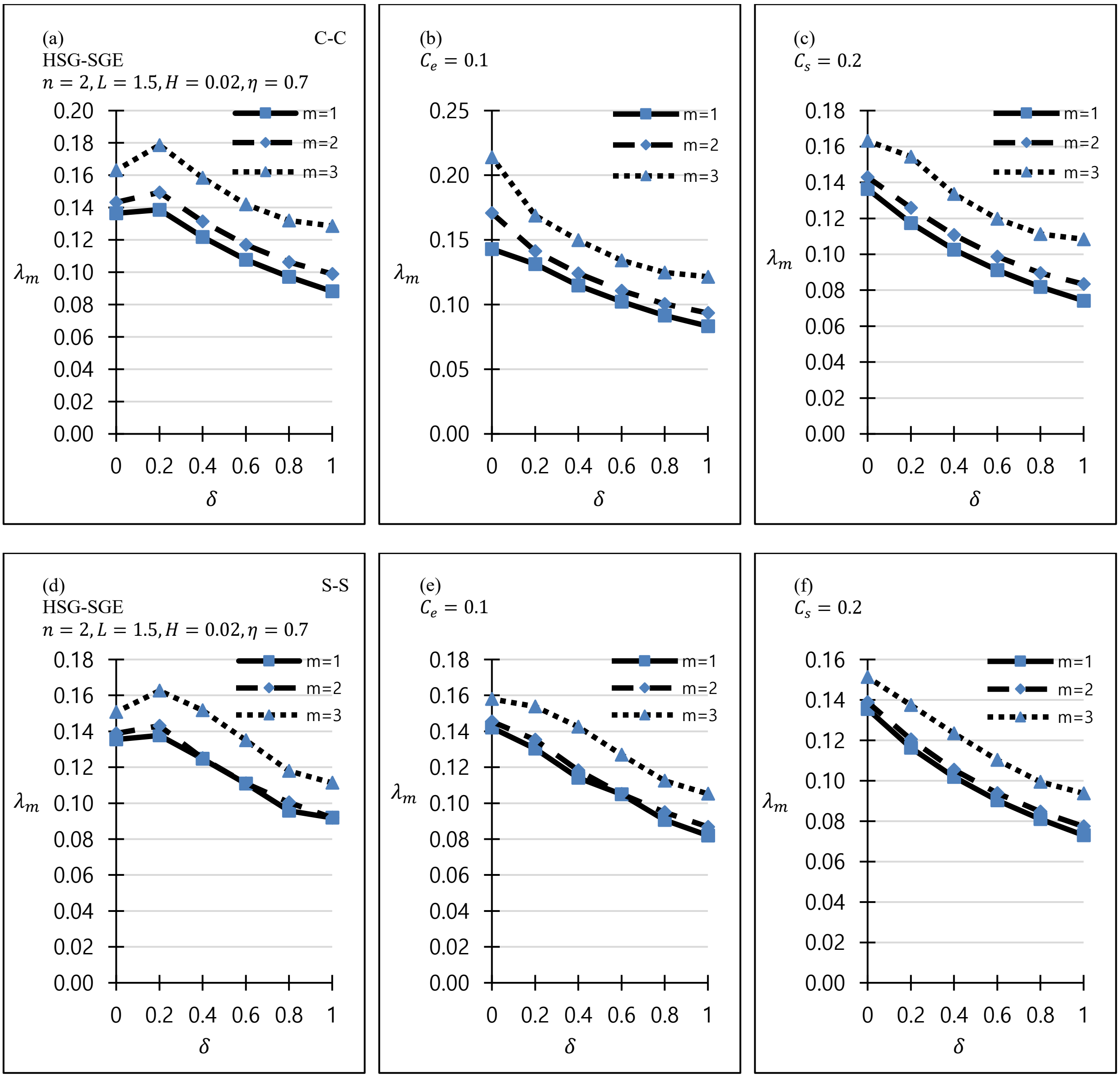

Fig. 1 Variation of frequency parameter λ_m $(m = 1,2,3)$ with relative layer thickness for two layered shells C–C and S-S boundary conditions

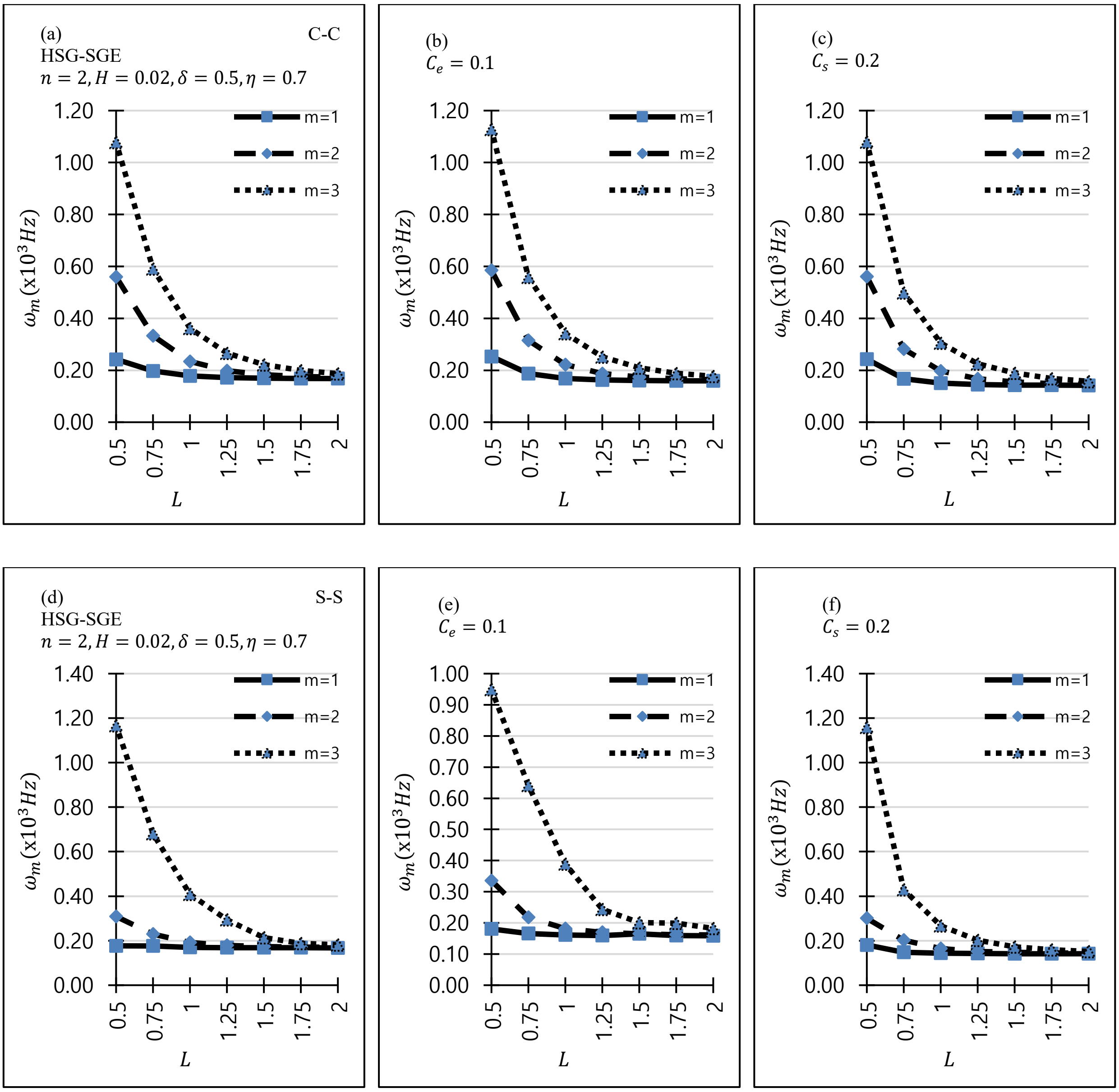

Fig 2 Effect of length of the shell on frequency parameter $\omega_m(\text{x}10^3 Hz)$ for different types of variation in thickness of layers for two layered shells under C–C and S-S boundary conditions

Three types of variable thickness namely linear, exponential and sinusoidal were investigated on the vibrational behavior of two layered cylindrical shell with fluid. The influence of linear (η=0.75), exponential (C_e=0.2) and sinusoidal (C_s=0.25) variations in thickness of layers on frequency parameters under C–C boundary conditions are shown in Table 1, Table 2 and Table 3, respectively. The values of $H = 0.02$, $L = 1.5$, $\delta = 0.5$ and $n = 2,4$ are fixed.

The study was extended by considering the shell under S-S boundary conditions. The study on the effect of linear (η=0.75), exponential (C_e=0.2) and sinusoidal (C_s=0.25) variations in thickness of layers on frequency parameters is depicted in Table 4, Table 5 and Table 6, respectively. The values of $H = 0.02$, $L = 1.5$, $\delta = 0.5$ and $n = 2,4$ are fixed.

Based on the results, the variable thickness has significant effect on the frequencies of the shell. For linear variation, the frequencies increase in the range $0.5 \leq \eta \leq 0.7$ and then it slightly decreases. For exponential variation, the values of frequencies decrease from $C_e =$ -0.2 to $C_e =$ -0.1 and then it slightly increases. Meanwhile, the values of $\lambda_m(m = 1,2,3)$ for the sinusoidal variation are almost constant. In general, results showed that the values of $\lambda_m(m = 1,2,3)$ for n=4 are higher compared to the values of $\lambda_m(m = 1,2,3)$ for n=2. Overall, the values of $\lambda_m(m = 1,2,3)$ under C-C boundary conditions are higher compared to those values of $\lambda_m(m = 1,2,3)$ under S-S boundary conditions.

Table 1 Variation of frequency parameters λ_m $(m = 1,2,3)$ with respect to the thickness parameter η under C-C boundary conditions for two layered shells

$H = 0.02, \quad \delta = 0.5, \quad L = 1.5$

η	$n = 2$			$n = 4$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0.5	0.096585	0.104412	0.126523	0.138833	0.150811	0.182922
0.7	0.114168	0.123721	0.149532	0.163594	0.177454	0.214732
0.9	0.101595	0.109693	0.132521	0.145762	0.158161	0.191305
1.1	0.092041	0.099566	0.120274	0.132374	0.144046	0.174213
1.3	0.084924	0.091815	0.11091	0.122492	0.133148	0.161032
1.5	0.079239	0.085639	0.103457	0.114535	0.124411	0.150475
1.7	0.07456	0.080574	0.097349	0.10794	0.11721	0.141784
1.9	0.070621	0.076324	0.09223	0.102353	0.111148	0.134477
2.1	0.067244	0.072696	0.087866	0.097543	0.105957	0.128224

Table 2 Variation of frequency parameters $\lambda_m (m = 1,2,3)$ with respect to the thickness parameter C_e under C-C boundary conditions for two layered shells

$H = 0.02, \quad \delta = 0.5, \quad L = 1.5$

C_e	$n = 2$			$n = 4$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
-0.2	0.086554	0.093604	0.113081	0.124762	0.135671	0.164096
-0.1	0.082712	0.089431	0.108046	0.119403	0.129782	0.156982
0	0.096555	0.104261	0.12595	0.138309	0.150608	0.182157
0.1	0.10805	0.117018	0.141361	0.155122	0.168352	0.205625
0.2	0.118416	0.12832	0.15508	0.16942	0.18371	0.222289

Table 3 Variation of frequency parameters $\lambda_m (m = 1,2,3)$ with respect to the thickness parameter C_s under C-C boundary conditions for two layered shells

$H = 0.02, \quad \delta = 0.5, \quad L = 1.5$

C_s	$n = 2$			$n = 4$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
-0.5	0.096677	0.104605	0.12648	0.138974	0.151083	0.182909
-0.4	0.096602	0.104057	0.126286	0.138861	0.150897	0.182624
-0.3	0.096582	0.104393	0.126319	0.138828	0.150338	0.182123
-0.2	0.096568	0.10433	0.126205	0.138805	0.150686	0.182278
-0.1	0.096164	0.104224	0.125977	0.138787	0.15055	0.18219
0	0.096555	0.104261	0.12595	0.138309	0.150608	0.182157
0.1	0.096473	0.104297	0.125988	0.138609	0.150657	0.182205
0.2	0.096558	0.104335	0.128125	0.138794	0.150716	0.182293
0.3	0.096614	0.104406	0.126153	0.138883	0.150804	0.182438
0.4	0.096646	0.104493	0.126299	0.138929	0.150926	0.182645
0.5	0.096677	0.104605	0.12648	0.138974	0.151083	0.182909

Table 4 Variation of frequency parameters $\lambda_m\ (m = 1,2,3)$ with respect to the thickness parameter η under S-S boundary conditions for two layered shells

$H = 0.02, \quad \delta = 0.5, \quad L = 1.5$

η	$n = 2$			$n = 4$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
0.5	0.095324	0.098769	0.134802	0.136534	0.142886	0.166433
0.7	0.113475	0.117541	0.144282	0.161378	0.168945	0.205873
0.9	0.10063	0.104272	0.125936	0.143872	0.150654	0.181005
1.1	0.091355	0.09469	0.11315	0.13108	0.137276	0.163357
1.3	0.084246	0.087351	0.103583	0.121195	0.126968	0.149934
1.5	0.078574	0.08152	0.09607	0.113267	0.118714	0.139365
1.7	0.073909	0.076719	0.089325	0.106725	0.111888	0.130734
1.9	0.069987	0.072696	0.084924	0.1012	0.106155	0.123541
2.1	0.066627	0.069258	0.080648	0.096458	0.101231	0.117286

Table 5 Variation of frequency parameters $\lambda_m\ (m = 1,2,3)$ with respect to the thickness parameter C_e under S-S boundary conditions for two layered shells

$H = 0.02, \quad \delta = 0.5, \quad L = 1.5$

C_e	$n = 2$			$n = 4$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
-0.2	0.085876	0.089042	0.105782	0.123464	0.129365	0.153051
-0.1	0.082039	0.085096	0.100674	0.11812	0.123794	0.145867
0	0.095656	0.099122	0.119033	0.137027	0.143501	0.171503
0.1	0.110925	0.111199	0.135419	0.153058	0.160234	0.193924
0.2	0.121682	0.1219	0.150705	0.167091	0.174851	0.214148

Table 6 Variation of frequency parameters $\lambda_m\ (m = 1,2,3)$ with respect to the thickness parameter C_s under S-S boundary conditions for two layered shells

$H = 0.02, \quad \delta = 0.5, \quad L = 1.5$

C_s	$n = 2$			$n = 4$		
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
-0.5	0.095764	0.099696	0.115452	0.137472	0.144415	0.16668
-0.4	0.095204	0.098662	0.116731	0.136361	0.142775	0.166477
-0.3	0.095349	0.098763	0.115404	0.136573	0.142995	0.166494
-0.2	0.095477	0.098898	0.115532	0.136745	0.143127	0.177128
-0.1	0.095584	0.099022	0.121718	0.136898	0.14332	0.174865
0	0.095656	0.099122	0.119033	0.137027	0.143501	0.171503
0.1	0.095701	0.099229	0.117613	0.137136	0.143279	0.169578
0.2	0.095726	0.099338	0.116685	0.137235	0.143863	0.168329
0.3	0.095744	0.09946	0.116078	0.137325	0.144032	0.167493
0.4	0.095753	0.099579	0.11567	0.137401	0.14412	0.166882
0.5	0.095764	0.099696	0.115452	0.137472	0.144415	0.16668

5. CONCLUSION

This study analyses the effect of thickness variation of cylindrical shells filled with fluid on frequencies. Based on the results, thickness variation (linear, exponential, sinusoidal) of the cylindrical shells significantly affects the frequencies. It can be highlighted that S-S boundary conditions give lower frequencies compared to C-C boundary conditions. An increase in the length of the cylinder leads to a decrease frequency of the shells. The findings of this study can be used to optimise the design of cylindrical shells by tailoring the variable thickness to achieve desired vibration characteristics. In addition, the effect of fluid interaction on shells, shell geometry, material properties and boundary conditions cannot be ignored as these factors significantly influence the vibrational behaviour.

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